arXiv:0807.2305v1 [hep-th] 15 Jul 2008

Recursion Relation for the Feynman Diagrams of the Effective Action for the Third Legendre Transformation

Chungku Kim

(Dated: July 15, 2008)

We derive a recursion relation of the Feynman diagrams of the effective action for the third Legendre transformation in case of the bosonic field theory with cubic interaction. We apply the recursion relation to obtain the Feynman diagrams of the effective action for the third Legendre transformation up to the five-loop order. The three-particle irreducibility of the Feynman diagrams of the effective action for the third Legendre transformation is shown by induction.

PACS numbers: 11.15.Bt, 12.38.Bx

I. INTRODUCTION

The effective action plays an important role in studies of the vacuum instability, dynamical symmetry breaking and the dynamics of composite particles[1] for a given particle physics model. The selective resummation of the effective action is important in the investigation of the equilibrium and the non-equilibrium dynamics of the quantum field theory [2]. The CJT effective action [3] which contains only two-particle-irreducible (2PI) Feynman diagrams [4] was widely used among several resummation schemes. Recently, the nPI effective action [2, 5] defined by the n-th Legendre transformation of the generating functional was studied extensively as a generalization of the CJT effective action which was obtained from the second Legendre transformation of the generating functional. Especially, the effective action for the third Legendre transformation was used in the investigation of the QED electrical conductivity [6]. However, it was shown that all the fourth Legendre transformation known yet actually contains the four-particle-reducible Feynman diagrams [7].

The recursive generation of the Feynman diagrams was investigated by using the functional integral identities $\int D\Phi \; \frac{\delta}{\delta\Phi}F[\Phi] = 0$ in case of the connected and the one-particle-irreducible (1PI) effective action was obtained for the multicomponent ϕ^4-theory, QED and scalar QED theories [8, 9, 10, 11, 12, 13, 14]. Recently, we have derived a new method to obtain the recursion relation for the ordinary [15] as well as CJT effective action [16] by using the functional derivative identities. In this paper, we apply this method to the Feynman diagrams of the effective action for the third Legendre transformation in case of the bosonic field theory with the cubic interaction. In Sec.II, we derive the recursion relation for the Feynman diagrams of the effective action for the third Legendre transformation. By using the recursion relation, we obtain the Feynman diagrams up to the five-loop order. Then we show the three-particle-irreducibility of the Feynman diagrams of the effective action for the third Legendre transformation by induction. In Sec.III, we give some discussion and conclusions.

II. RECURSION RELATION FOR THE FEYNMAN DIAGRAMS OF THE EFFECTIVE ACTION FOR THE THIRD LEGENDRE TRANSFORMATION

In this section, we will first derive a recursion relation for the Feynman Diagrams of the effective action for the third Legendre transformation for the bosonic field theory with the cubic interaction. The classical action is given by

$$S[\Phi] = \frac{1}{2}\Phi_A D^{-1}_{0AB}\Phi_B + \frac{1}{6}g_{ABC}\Phi_A\Phi_B\Phi_C. \tag{1}$$

In this paper, we use a notation in which the capital letters contain both the space-time variables and the internal indices and repeated capital letters mean both integrations over continuous variables and sums over internal indices. For example, if the capital letter A contains a space-time variable x and the internal index i,

$$J_A\Phi_A \equiv \sum_i \int d^4x J_i(x)\Phi_i(x). \tag{2}$$

The generating functional $W[J_1, J_2, J_3]$ is given by

$$W[J_1, J_2, J_3] = -\hbar \ln \int D\Phi \; Exp[-\frac{1}{\hbar}(S[\Phi] + J_{1A}\Phi_A + \frac{1}{2!}J_{2AB}\Phi_A\Phi_B + \frac{1}{3!}J_{3ABC}\Phi_A\Phi_B\Phi_C)], \tag{3}$$

where the external source J_{2AB} and J_{3ABC} is symmetric under an exchange of the indices. The functional derivatives of the $W[J_1, J_2, J_3]$ with respect to the external sources are given by the classical field ϕ, the full propagator G and the proper three-vertex V_3 as

$$\frac{\delta W[J_1,J_2,J_3]}{\delta J_{1A}} = \langle \Phi_A \rangle = \phi_A, \tag{4}$$

$$2!\frac{\delta W[J_1,J_2,J_3]}{\delta J_{2AB}} = \langle \Phi_A \Phi_B \rangle = \phi_A\phi_B + \hbar G_{AB}, \tag{5}$$

$$3!\frac{\delta W[J_1,J_2,J_3]}{\delta J_{3ABC}} = \langle \Phi_A \Phi_B \Phi_B \rangle = \phi_A\phi_B\phi_C + \hbar(\phi_A G_{BC} + \phi_B G_{AC} + \phi_C G_{AB}) - \hbar^2 G_{AA'}G_{BB'}G_{CC'}V_{3A'B'C'}. \tag{6}$$

From Eqs.(3),(4) ,(5)and (6) we can see that

$$G_{AB} \equiv -\frac{\delta^2 W[J_1,J_2,J_3]}{\delta J_{1A}\delta J_{1B}}. \tag{7}$$

and

$$G_{AA'}G_{BB'}G_{CC'}V_{3A'B'C'} \equiv -\frac{\delta^2 W[J_1,J_2,J_3]}{\delta J_{1A}\delta J_{1B}\delta J_{1C}}. \tag{8}$$

By inverting Eqs.(4),(5) and (6), one can obtain the functionals $J_i[\phi, G, V_3]$ $(i = 1, 2, 3)$. Then, the effective action for the third Legendre transformation is defined as

$$\begin{aligned}\Gamma[\phi, G, V_3] &= W[J_1,J_2,J_3] - J_{1A}\phi_A - \frac{1}{2}J_{2AB}(\phi_A\phi_B + \hbar G_{AB}) \\ &\quad -\frac{1}{6}J_{3ABC}\{\phi_A\phi_B\phi_C + \hbar(\phi_A G_{BC} + \phi_B G_{AC} + \phi_C G_{AB}) - \hbar^2 G_{AA'}G_{BB'}G_{CC'}V_{3A'B'C'}\}.\end{aligned} \tag{9}$$

From Eqs.(4), (5),(6) and (9), one can obtain the following relations:

$$\frac{\delta\Gamma[\phi,G,V_3]}{\delta\phi_A} = -J_{1A} - J_{2AB}\phi_B - \frac{1}{2}J_{3ABC}(\phi_B\phi_C + \hbar G_{BC}), \tag{10}$$

$$\frac{\delta\Gamma[\phi,G,V_3]}{\delta G_{AB}} = -\frac{\hbar}{2}J_{2AB} - \frac{\hbar}{2}J_{3ABC}\phi_C + \frac{\hbar^2}{4}(J_{3ACD}V_{3BC'D'} + J_{3BCD}V_{3AC'D'})G_{CC'}G_{DD'}, \tag{11}$$

$$\frac{\delta\Gamma[\phi,G,V_3]}{\delta V_{3ABC}} = \frac{\hbar^2}{6}G_{AA'}G_{BB'}G_{CC'}J_{3A'B'C'}. \tag{12}$$

Also, from Eqs.(3) and (9), we obtain

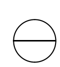
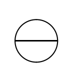
$$\begin{aligned}\exp\{-\frac{1}{\hbar}\Gamma[\phi,G,V_3]\} &= \int D\Phi\ \exp\{-\frac{1}{\hbar}(S(\Phi) + J_{1A}(\Phi_A - \phi_A) + \frac{1}{2}J_{2AB}(\Phi_A\Phi_B - \phi_A\phi_B - \hbar G_{AB}) + \frac{1}{6}J_{3ABC}(\Phi_A\Phi_B\Phi_C \\ &\quad -\phi_A\phi_B\phi_C - \hbar(\phi_A G_{BC} + \phi_B G_{AC} + \phi_C G_{AB}) + \hbar^2 G_{AA'}G_{BB'}G_{CC'}V_{3A'B'C'})\}.\end{aligned} \tag{13}$$

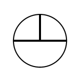
By expanding the effective action $\Gamma[\phi, G, V_3]$ around $\hbar$, we can obtain the loop-wise expansion of $\Gamma[\phi, G, V_3]$[17] as

$$\Gamma[\phi,G,V_3] = \sum_{l=0}\hbar^l\Gamma^{(l)}[\phi,G,V_3] \tag{14}$$

Recently Berges[2] has obtained $\Gamma^{(l)}[\phi, G, V_3]$ up to three loop order by applying the equivalence hierarchy principle to the previously known Feynman diagrams for the 2PI effective action;

$$\Gamma^{(0)}[\phi,G] = S[\phi],\ \Gamma^{(1)}[\phi,G] = \frac{1}{2}Tr\ln G^{-1} - \frac{1}{2}TrG(G^{-1} - D^{-1}), \tag{15}$$

$$\Gamma^{(2)}[\phi,G,J_3] = \frac{1}{12}V_{3ABC}V_{3PQR}G_{AP}G_{BQ}G_{CR} - \frac{1}{6}V_{3ABC}\ g_{PQR}G_{AP}G_{BQ}G_{CR} = \frac{1}{12}$$  $$-\frac{1}{6}$$  g , (16)

$$\Gamma^{(3)}[\phi,G,J_3] = -\frac{1}{24}V_{3ABC}V_{3DEF}V_{3PQR}V_{3STU}G_{AD}G_{BP}G_{CS}G_{DQ}G_{ET}G_{RU} = -\frac{1}{24}$$  (17)

where

$$D^{-1} \equiv \frac{\delta^2 S[\phi]}{\delta\phi_A \delta\phi_B} = D^{-1}_{0AB} + g_{ABC}\phi_C, \tag{18}$$

and we have used the graphical representation in which a line and a three point vertex represents the propagator G and the proper three-vertex V_{3ABC} respectively and the three point vertex with the letter g means the three point vertex g_{ABC}. In the Appendix, it is shown that $J_3[\phi, G, V_3]$ is actually independent of ϕ and hence the Feynman diagrams of the $\Gamma^{(l)}[\phi, G, V_3](l \geq 2)$ is independent of ϕ. Then from (12), we can see that only $\Gamma^{(0)}$ and $\Gamma^{(1)}$ can contribute to $\frac{\delta\Gamma}{\delta\phi_C}$. Then from (15) and (18) we obtain

$$\frac{\delta^2\Gamma}{\delta\phi_A \delta G_{BC}} = \frac{\hbar}{2} g_{ABC}, \tag{19}$$

By taking the derivative $\frac{\delta}{\delta\phi_C}$ to (11), we obtain

$$\frac{\delta^2\Gamma}{\delta\phi_C \delta G_{AB}} = -\ \frac{\hbar}{2}\frac{\delta J_{2AB}}{\delta\phi_C} -\ \frac{\hbar}{2} J_{3ABC} \tag{20}$$

By comparing (19) and (20), we obtain

$$\frac{\delta J_{2AB}}{\delta\phi_C} = -J_{3ABC} - g_{ABC} \tag{21}$$

Then, since J_3 is independent of ϕ, we can write $J_2[\phi, G, V_3]$ as

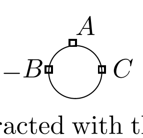
$$J_2[\phi, G, V_3]_{AB} = -(J_3[G, V_3]_{ABC} + g_{ABC})\phi_C + K_2[G, V_3]_{AB} \tag{22}$$

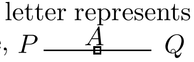
where K_2 is the ϕ independent part of J_2. In terms of K_2, we can write (11) and

$$\frac{\delta\Gamma[\phi, G, V_3]}{\delta G_{AB}} = -\ \frac{\hbar}{2} g_{ABC}\phi_C -\ \frac{\hbar}{2} K_{2AB} +\ \frac{\hbar^2}{4}(J_{3ACD}V_{3BC'D'} + J_{3BCD}V_{3AC'D'})G_{CC'}G_{DD'}, \tag{23}$$

From Eqs.(12),(15) and (16), we obtain

$$J^{(0)}_{3ABC} = -g_{ABC} + V_{3ABC},\ J^{(1)}_{3ABC} = V_{3ADE}V_{3BPQ}V_{3CST}G_{DP}G_{QS}G_{TE} = - \tag{24}$$



where a box with an capital letter represents the vertex which have indices that is not contracted with the propagators attached to it. For example,  means $V_{3AP'Q'}G_{P'P}G_{Q'Q}$. From (15),(23) and (24), we obtain

$$K^{(0)}_{2AB} = G^{-1}_{AB} - D^{-1}_{0AB},\ J^{(0)}_{2AB} = -(g_{ABC} + J^{(0)}_{3ABC})\phi_C + K^{(0)}_{2AB} = -V_{3ABC}\phi_C + G^{-1}_{AB} - D^{-1}_{0AB} \tag{25}$$

Now consider the functional identities satisfied by the two sources $J_1[\phi, G, V_3]$ and $J_3[G, V_3]$

$$\frac{\delta J_{1A}}{\delta\phi_P}\frac{\delta\phi_P}{\delta J_{1B}} + \frac{\delta J_{1A}}{\delta G_{PQ}}\frac{\delta G_{PQ}}{\delta J_{1B}} + \frac{\delta J_{1A}}{\delta V_{3PQR}}\frac{\delta V_{3PQR}}{\delta J_{1B}} = \delta_{AB} \tag{26}$$

and

$$\frac{\delta J_{3ACD}}{\delta G_{PQ}}\frac{\delta G_{PQ}}{\delta J_{1B}} + \frac{\delta J_{3ACD}}{\delta V_{3PQR}}\frac{\delta V_{3PQR}}{\delta J_{1B}} = 0 \tag{27}$$

where we have used the fact that $\frac{\delta J_{3ACD}}{\delta\phi_P} = 0$. By eliminating the term $\frac{\delta V_{3PQRD}}{\delta J_{1B}}$ from Eqs.(26) and (27), we obtain

$$\frac{\delta J_{1A}}{\delta\phi_P}\frac{\delta\phi_P}{\delta J_{1B}} + \frac{\delta J_{1A}}{\delta G_{PQ}}\frac{\delta G_{PQ}}{\delta J_{1B}} - \frac{\delta J_{1A}}{\delta V_{3PQR}}\Omega^{-1}_{PQR,CDE}\frac{\delta J_{3CDE}}{\delta G_{PQ}}\frac{\delta G_{PQ}}{\delta J_{1B}} = \delta_{AB}, \tag{28}$$

where

$$\Omega_{ABC,DEF} \equiv \frac{\delta J_{3ABC}}{\delta V_{3DEF}} = \frac{6}{\hbar^2}G^{-1}_{AP}G^{-1}_{BQ}G^{-1}_{APCR}\frac{\delta^2\Gamma}{\delta V_{3DEF}\delta V_{3PQR}}. \tag{29}$$

From (24) and (29), we obtain

$$\Omega^{(0)}_{ABC,DEF} = \frac{1}{6}(\delta_{AD}\delta_{BE}\delta_{CR} + \delta_{AE}\delta_{BR}\delta_{CD} + \delta_{AR}\delta_{BD}\delta_{CE} + \delta_{AD}\delta_{BR}\delta_{CE} + \delta_{AR}\delta_{BE}\delta_{CD} + \delta_{AE}\delta_{BD}\delta_{CR}). \tag{30}$$

From Eqs.(4),(7) and (8), we obtain

$$\frac{\delta\phi_A}{\delta J_{1B}} = \frac{\delta^2 W[J]}{\delta J_C \delta J_B} = -G_{CB}, \tag{31}$$

$$\frac{\delta G_{AB}}{\delta J_{1C}} = -\frac{\delta^2 W[J]}{\delta J_{1A}\delta J_{1B}\delta J_{1C}} = G_{AA'}G_{BB'}G_{CC'}V_{3A'B'C'} \tag{32}$$

Also from Eqs.(10),(15),(19) and the fact that $\frac{\delta J_{3ABC}}{\delta \phi_P} = 0$, we obtain

$$\frac{\delta J_{1A}}{\delta\phi_P} = -D^{-1}_{0AB} - J_{2AP}, \tag{33}$$

$$\frac{\delta J_{1A}}{\delta G_{PQ}} = -\frac{\hbar}{2}g_{APQ} - \frac{\delta J_{2AC}}{\delta G_{PQ}}\phi_C - \frac{\hbar}{2}J_{3APQ} - \frac{1}{2}(\phi_B\phi_C + \hbar G_{BC})\frac{\delta J_{3ABC}}{\delta G_{PQ}}, \tag{34}$$

$$\frac{\delta J_{1A}}{\delta V_{3PQR}} = -\frac{\delta J_{2AC}}{\delta V_{3PQR}}\phi_C - \frac{1}{2}(\phi_B\phi_C + \hbar G_{BC})\frac{\delta J_{3ABC}}{\delta V_{3PQR}}. \tag{35}$$

By substituting Eqs.(31),(32),(33),(34) and (35) into (28), we obtain

$$\begin{aligned}\delta_{AB} &= (D^{-1}_{0AP} + J_{2AP})G_{PB} + [\{-\frac{\hbar}{2}g_{APQ} - \frac{\delta J_{2AC}}{\delta G_{PQ}}\phi_C - \frac{\hbar}{2}J_{3APQ} - \frac{1}{2}(\phi_C\phi_{C'} + \hbar G_{CC'})\frac{\delta J_{3ACC'}}{\delta G_{PQ}}\} \\ &\quad +\{\frac{\delta J_{2AC}}{\delta V_{3RST}}\phi_C + \frac{1}{2}(\phi_C\phi_{C'} + \hbar G_{CC'})\frac{\delta J_{3ACC'}}{\delta V_{3RST}}\}\Omega^{-1}_{RST,DEF}\frac{\delta J_{3DEF}}{\delta G_{PQ}}]\left(G^3V_3\right)_{BPQ} \\ &= (D^{-1}_{0AP} + J_{2AP})G_{PB} + [\{-\frac{\hbar}{2}g_{APQ} - \frac{\delta J_{2AC}}{\delta G_{PQ}}\phi_C - \frac{\hbar}{2}J_{3APQ}\} + \phi_C\frac{\delta J_{2AC}}{\delta V_{3RST}}\Omega^{-1}_{RST,DEF}\frac{\delta J_{3DEF}}{\delta G_{PQ}}]\left(G^3V_3\right)_{BPQ}\end{aligned} \tag{36}$$

where $\left(G^3V_3\right)_{BPQ} \equiv G_{BB'}G_{PP'}G_{QQ'}V_{3B'P'Q'}$ and we have used (29) to obtain the last line of above equation. Note that in the last line of the above equation, although there is a terms proportional to ϕ^2 , they cancel each other (see (22)). As a result, one can obtain two independent equations by extracting terms proportional to ϕ and those independent of ϕ respectively. Among these two equations, the one proportional to ϕ_C can be obtained by using (12) and (22) as

$$\begin{aligned}0 &= (-g_{APC} - J_{3APC})G_{PB} + [-\frac{\delta K_{2AC}}{\delta G_{PQ}} + \frac{\delta K_{2AC}}{\delta V_{3RST}}\Omega^{-1}_{RST,DEF}\frac{\delta J_{3DEF}}{\delta G_{PQ}}]\left(G^3V_3\right)_{BPQ}, \\ &= (-g_{APC} - J_{3APC})G_{PB} + [\frac{2}{\hbar}\frac{\delta^2\Gamma[\phi,G,V_3]}{\delta G_{AC}\delta G_{PQ}} - \hbar(V_{3APS}J_{3CQT} +_{3AQS} J_{3CPT})G_{ST} \\ &\quad -\frac{\hbar}{3}G_{RR'}G_{SS'}G_{TT'}\frac{\delta J_{3R'S'T'}}{\delta G_{AC}}\Omega^{-1}_{RST,DEF}\frac{\delta J_{3DEF}}{\delta G_{PQ}}]\left(G^3V_3\right)_{BPQ}.\end{aligned} \tag{37}$$

where we have used (23) to obtain the last line of the above equation. Then, by multiplying $\frac{\hbar^2}{6}V_{3SBT}G_{AS}G_{CT}$ and by using (12), we can obtain

$$\begin{aligned}V_{3SBT}\frac{\delta\Gamma[\phi,G,V_3]}{\delta V_{3SBT}} &= \frac{1}{3}[\,\hbar\frac{\delta^2\Gamma[\phi,G,V_3]}{\delta G_{AC}\delta G_{PQ}} - \frac{\hbar^3}{6}G_{RR'}G_{SS'}G_{TT'}\frac{\delta J_{3R'S'T'}}{\delta G_{AC}}\Omega^{-1}_{RST,DEF}\frac{\delta J_{3DEF}}{\delta G_{PQ}} - \frac{\hbar^3}{2}(V_{3APS}J_{3CQT} \\ &\quad +V_{3AQS}J_{3CPT})G_{ST}]\left(G^2V_3\right)_{ACB}G_{BB'}\left(G^2V_3\right)_{B'PQ} - \frac{\hbar^2}{2}V_{3ABC}\ g_{PQR}G_{AP}G_{BQ}G_{CR} \\ &= \frac{1}{3}[\,\hbar\frac{\delta^2\Gamma[\phi,G,V_3]}{\delta G_{AC}\delta G_{PQ}} - \frac{\hbar^3}{6}G_{RR'}G_{SS'}G_{TT'}\frac{\delta J_{3R'S'T'}}{\delta G_{AC}}\Omega^{-1}_{RST,DEF}\frac{\delta J_{3DEF}}{\delta G_{PQ}}]\left(G^2V_3\right)_{ACB}G_{BB'}\left(G^2V_3\right)_{B'PQ} \\ &\quad -2\hbar\frac{\delta\Gamma[\phi,G,V_3]}{\delta V_{3CSQ}}V_{3APS}G_{AA'}V_{3A'CB}G_{BB'}G_{PP'}V_{3B'P'Q} - \frac{\hbar^2}{2}V_{3ABC}\ g_{PQR}G_{AP}G_{BQ}G_{CR}\end{aligned} \tag{38}$$

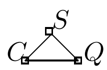
where $\left(G^2V_3\right)_{ABC} \equiv V_{3AB'C'}G_{B'B}G_{C'C}$. Note that the operation $V_{3SBT}\frac{\delta\Gamma[\phi,G,V_3]}{\delta V_{3SBT}}$ is equivalent to multiplying each Feynman diagrams in Γ by N_3 which is the number of the vertex V_3. By using the fact that the number of the propagator L is related as $3N_3 = 2L$ and that the Euler formula for the number of the loop l is given by $l = L - N_3 + 1$, we obtain $N_3 = 2l - 2$. Then by using (14), we get

$$V_{3SBT}\frac{\delta\Gamma[\phi,G,V_3]}{\delta V_{3SBT}} = \sum_{l=0}(2l-2)\hbar^l\Gamma^{(l)}[\phi,G,V_3] \tag{39}$$

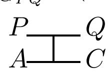
Since the last term on the right hand side (R.H.S.) of (38) contributes only to $\Gamma^{(2)}$, the $l-$th loop order effective action for the third Legendre transformation $\Gamma^{(l)}[\phi,G,V_3]$ in case of $l \geq 3$ is given by

$$\Gamma^{(l)}[\phi,G,V_3] = \frac{1}{6(l-1)}[-6\frac{\delta\Gamma^{(l-1)}[\phi,G,V_3]}{\delta V_{3CSQ}}V_{3APS}G_{AA'}V_{3A'CB}G_{BB'}G_{PP'}V_{3B'P'Q} \tag{40}$$
$$+\{\frac{\delta^2\Gamma^{(l-1)}[\phi,G,V_3]}{\delta G_{AC}\delta G_{PQ}} - \frac{1}{6}G_{RR'}G_{SS'}G_{TT'}\sum_{\substack{p\geq1,q\geq0,r\geq1\\ p+q+r=l-3}}\frac{\delta J^{(p)}_{3R'S'T'}}{\delta G_{AC}}\Omega^{-1(q)}_{RST,DEF}\frac{\delta J^{(r)}_{3DEF}}{\delta G_{PQ}}\}\left(G^2V_3\right)_{ACB}G_{BB'}\left(G^2V_3\right)_{B'PQ}]$$

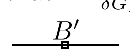
Equation (40) is the central result of this paper. Each term of this equation can be obtained as follows;

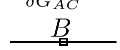
(i) In order to obtain $6\frac{\delta\Gamma^{(l-1)}[\phi,G,V_3]}{\delta V_{3CSQ}}V_{3APS}G_{AA'}V_{3A'CB}G_{BB'}G_{PP'}V_{3B'P'Q}$ we remove V_{3CSQ} and replace it with 

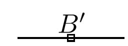
and then multiply the results by 6. The result is equivalent to connecting the two different propagators which are connected to the same three point vertex V_3 of $\Gamma^{(l-1)}[\phi,G,V_3]$ in all possible way and then multiply the results by 2.

(ii) In order to obtain $\frac{\delta^2\Gamma^{(l-1)}[\phi,G,V_3]}{\delta G_{AC}\delta G_{PQ}}\left(G^2V_3\right)_{ACB}G_{BB'}\left(G^2V_3\right)_{B'PQ}$ we remove the two propagators G_{AC} and G_{PQ} from $\Gamma^{(l-1)}$ and replace it with 

. The result is equivalent to connecting the two different propagators of $\Gamma^{(l-1)}[\phi,G,V_3]$ in all possible way and then multiply the results by 2.

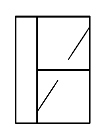
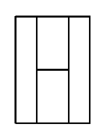
(iii)In case of the last term of (40), note that $\frac{\delta J^{(r)}_{3DEF}}{\delta G_{PQ}}\left(G^2V_3\right)_{B'PQ}$ corresponds to replacing one of the propagators of $J^{(q)}_{3DEF}$ with 

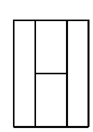
. Then, in order to obtain $G_{RR'}G_{SS'}G_{TT'}\sum_{p,q,ratopp+q+r=l-3}\frac{\delta J^{(p)}_{3R'S'T'}}{\delta G_{AC}}\Omega^{-1(q)}_{RST,DEF}\frac{\delta J^{(r)}_{3DEF}}{\delta G_{PQ}}\left(G^2V_3\right)_{ACB}G_{BB'}\left(G^2V_3\right)_{B'PQ}$, we replace one of the propagators of $J^{(p)}_{3R'S'T'}$ with 

and one of the propagators of $J^{(q)}_{3DEF}$ with 

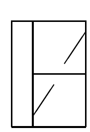
and connect the points B and B'. Then connect with $G_{RR'}G_{SS'}G_{TT'}\Omega^{-1(q)}_{RST,DEF}$. Note that since $\frac{\delta J^{(0)}_{3ABC}}{\delta G_{PQ}} = 0$, this term contributes to $\Gamma^{(l)}[\phi,G,V_3]$ when $l \geq 5$.

Now let us apply (40) to obtain the four and five loop Feynman diagrams of the effective action for the third order Legendre transformation. From (17) we obtain

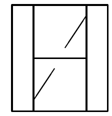
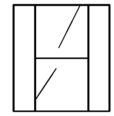
$$\frac{\delta^2\Gamma^{(3)}[\phi,G,V_3]}{\delta G_{AC}\delta G_{PQ}}\left(G^2V_3\right)_{ACB}G_{BB'}\left(G^2V_3\right)_{B'PQ} \;=\; -\frac{1}{4} \tag{41}$$

$-$


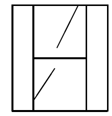
$$6\frac{\delta\Gamma^{(3)}[\phi,G,V_3]}{\delta V_{3CSQ}}V_{3APS}G_{AA'}V_{3A'CB}G_{BB'}G_{PP'}V_{3B'P'Q} \;=\; -2 \tag{42}$$


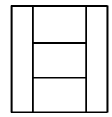
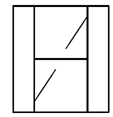
and by substituting these results to (40) we obtain

$$\Gamma^{(4)}[\phi,G,V_3] = -\frac{1}{72} \tag{43}$$


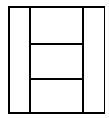
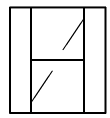
In case of $l=5$, there is a contribution from the third term of (40) so that

$$\frac{\delta^2\Gamma^{(4)}[\phi,G,V_3]}{\delta G_{AC}\delta G_{PQ}}\left(G^2V_3\right)_{ACB}G_{BB'}\left(G^2V_3\right)_{B'PQ} = -\frac{1}{2}$$  $$-\frac{1}{2}$$  (44)

$$6\frac{\delta\Gamma^{(4)}[\phi,G,V_3]}{\delta V_{3CSQ}}V_{3APS}G_{AA'}V_{3A'CB}G_{BB'}G_{PP'}V_{3B'P'Q} = -$$  (45)

$$G_{RR'}G_{SS'}G_{TT'}\frac{\delta J^{(1)}_{3R'S'T'}}{\delta G_{AC}}\Omega^{-1(0)}_{RST,DEF}\frac{\delta J^{(1)}_{3DEF}}{\delta G_{PQ}}\left(G^2V_3\right)_{ACB}G_{BB'}\left(G^2V_3\right)_{B'PQ} = 3$$  $$+6$$  (46)

and by substituting these results to (40) we obtain

$$\Gamma^{(5)}[\phi,G,V_3] = -\frac{1}{48}$$  $$-\frac{1}{16}$$  (47)

We can see that the Feynman diagrams of the $\Gamma^{(4)}[\phi,G,V_3]$ and $\Gamma^{(5)}[\phi,G,V_3]$ given in (43) and (47) coincide with the 3PI diagrams of the 2PI effective action[18].

Finally, let us show the three-particle-irreducibility of the Feynman diagrams of $\Gamma^{(l)}[\phi,G,V_3]$ by induction. For this purpose, assume that all the Feynman diagrams of the $\Gamma^{(k)}[\phi,G,V_3](k<l)$ are 3PI diagrams. Then as discussed in (i) and (ii), the sum of the first and the second term of R.H.S. of (40) is equal to connecting the two different propagators which are not connected to the same vertex V_3 of $\Gamma^{(l-1)}[\phi,G,V_3]$ in all possible way. Since the Feynman diagrams of the $\Gamma^{(k)}[\phi,G,V_3](k<l)$ are 3PI diagrams by assumption, the sum of the first and the second term of R.H.S. of (40) gives only 3PI Feynman diagrams (see (41),(42),(44) and (45)). Now, by using (30), the perturbative expansion of the $\Omega^{-1(q)}$ is given by

$$\Omega^{-1(q)}_{ABC,DEF} = -[\Omega^{(1)}_{ABC,STU}\Omega^{-1(q-1)}_{STU,DEF}+\Omega^{(2)}_{ABC,STU}\Omega^{-1(q-2)}_{STU,DEF}+...+\Omega^{(q)}_{ABC,STU}]. \tag{48}$$

and by repeated use of this equation, Ω^{-1} can be obtained as the sum of the products of Ω's so that

$$\Omega^{-1(q)}_{ABC,DEF} = \sum_{\substack{q_1,q_2,\dots,q_n\\ q_1+q_2+..+.q_n=q}} C^{(q)}_{q_1q_2...q_n}\Omega^{(q_1)}_{ABC,A_1B_1C_1}\Omega^{(q_2)}_{A_1B_1C_1,A_2B_2C_2}...\Omega^{(q_n)}_{A_{n-1}B_{n-1}C_{n-1},A_nB_nC_n} \tag{49}$$

Hence the third term of (40) is sum of terms of the form

$$G_{RR'}G_{SS'}G_{TT'}\frac{\delta J^{(p)}_{3R'S'T'}}{\delta G_{AC}}\Omega^{(q_1)}_{RST,R_1S_1T_1}\Omega^{(q_2)}_{R_1S_1T_1,R_2S_2T_2}...\Omega^{(q_n)}_{R_{n-1}S_{n-1}T_{n-1},R_nS_nT_n}\frac{\delta J^{(r)}_{3DEF}}{\delta G_{PQ}}\left(G^2V_3\right)_{ACB}G_{BB'}\left(G^2V_3\right)_{B'PQ} \tag{50}$$

with $q_1+q_2+...+q_n=q$. By noting that $\frac{\delta J^{(r)}_{3DEF}}{\delta G_{PQ}}\left(G^2V_3\right)_{B'PQ}$ corresponds to replacing one of the propagators of $J^{(q)}_{3DEF}$ with $\underline{\quad B' \quad}$, the graphical representation of (50) is given in Fig.1. Note that in Fig.1, part(a) is three-particle-irreducible since if we connect the points A,B and C of $J^{(k)}_{3ABC}$ to the vertex V_{3ABC}, we should obtain the 3PI diagram (see (12)). Similarly, part(b) is three-particle-irreducible since if we connect the points A,B and C of $\Omega^{(k)}_{ABC,DEF}$ with $A\,\underline{\;|^{B}\;}\,C$ and connect the points D,E and F to the vertex V_{3DEF} , we should obtain the 3PI diagram (see (29)). Other boxes and the oval have same property. Then, it is clear that Fig.1 is a 3PI diagram.

III. DISCUSSIONS AND CONCLUSIONS

In this paper, we have obtained the recursion relation for the effective action for the third Legendre transformation by using the functional derivative identity. We have applied the result to the case of Feynman diagrams up to the five-loop order for the bosonic field theory. Then we prove the three-particle-irreducibility of the Feynman diagrams of the effective action for the third Legendre transformation by using the recursion relation

Acknowledgments

This research was supported in part by the Institute of Natural Science.

IV. APPENDIX

In order to see that J_3 does not depend on ϕ , let us consider the perturbative derivation of $\Gamma^{(l)}[\phi, G, V_3]$ [15] and define $\overline{\Delta}$ as

$$\overline{\Delta}[\phi, G, J_3] = W[J_1, J_2, J_3] - J_{1A}\phi_A - \frac{1}{2}J_{2AB}(\phi_A\phi_B + \hbar G_{AB}) \tag{51}$$

Note that $\overline{\Delta}[\phi, G, J_3]$ is the 2PI effective action [3] with the classical action given by $\overline{S}[\Phi] = S[\Phi] + \frac{1}{6}J_{3ABC}\Phi_A\Phi_B\Phi_C$ so that the first three terms of perturbative expansion of $\overline{\Delta}$ is given by

$$\begin{aligned} \overline{\Delta}^{(0)}[\phi, G, J_3] &= \overline{S}[\phi] = S[\phi] + \frac{1}{6}J_{3ABC}\phi_A\phi_B\phi_C, \ \overline{\Delta}^{(1)}[\phi, G, J_3] = \frac{1}{2}Tr\ln G^{-1} - \frac{1}{2}TrG(G^{-1} - \overline{D}^{-1}), & (52) \\ \overline{\Delta}^{(2)}[\phi, G, J_3] &= -\frac{1}{12}(g_{ABC} + J_{3ABC})(g_{PQR} + J_{3PQR})G_{AP}G_{BQ}G_{CR} & (53) \end{aligned}$$

where

$$\overline{D}^{-1}_{AB}[\phi] \equiv \frac{\delta^2\overline{S}[\phi]}{\delta\phi_A\delta\phi_B} = D^{-1}_{AB}[\phi] + \frac{1}{2}J_{3ABC}\phi_C \tag{54}$$

with $D^{-1}_{AB}[\phi] \equiv \frac{\delta^2 S[\phi]}{\delta\phi_A\delta\phi_B}$. The higher order terms $\overline{\Delta}^{(k)}(k \succeq 2)$ is composed of 2PI vacuum diagrams with the propagator G and the three point vertex $g_{ABC} + J_{3ABC}$. It follows that $\overline{\Delta}^{(k)}(k \succeq 2)$ does not depend on ϕ. Next let us define Δ as

$$\Delta[\phi, G, J_3] = \overline{\Delta}[\phi, G, J_3] - \frac{1}{6}J_{3ABC}[\phi_A\phi_B\phi_C + \hbar(\phi_A G_{BC} + \phi_B G_{AC} + \phi_C G_{AB})] \tag{55}$$

The perturbative expansion of Δ is given by

$$\Delta^{(0)}[\phi, G, J_3] = S[\phi], \ \Delta^{(1)}[\phi, G, J_3] = \frac{1}{2}Tr\ln G^{-1} - \frac{1}{2}TrG(G^{-1} - D^{-1}), \tag{56}$$

and $\Delta^{(k)} = \overline{\Delta}^{(k)}(k \succeq 2)$ so that $\Delta^{(k)}(k \succeq 2)$ does not depend on ϕ. From (3),(9),(51) and (55), we can see that $\Gamma[\phi, G, V_3]$ can be obtained from $\Delta[\phi, G, J_3]$ by the Legendre transformation with respect to J_3 as

$$\Gamma[\phi, G, V_3] = \Delta[\phi, G, J_3] + \frac{\hbar^2}{6}J_{3ABC}G_{AA'}G_{BB'}G_{CC'}V_{3A'B'C'} \tag{57}$$

In order to determine $J_3[\phi, G, V_3]$, let us take the derivative of (57) with respect to V_3 as

$$\frac{\delta\Gamma[\phi, G, V_3]}{\delta V_{3PQR}} = (\frac{\delta\Delta[\phi, G, J_3]}{\delta J_{3ABC}} + \frac{\hbar^2}{6}G_{AA'}G_{BB'}G_{CC'}V_{3A'B'C'})\frac{\delta J_{3ABC}}{\delta V_{3PQR}} + \frac{\hbar^2}{6}J_{3ABC}G_{AP}G_{BQ}G_{CR} \tag{58}$$

By comparing (58) with (12) we obtain

$$\frac{\delta\Delta[\phi, G, J_3]}{\delta J_{3ABC}} = -\frac{\hbar^2}{6}G_{AA'}G_{BB'}G_{CC'}V_{3A'B'C'} \tag{59}$$

and by using this, we can obtain the perturbative expansion of the $J_3 = \sum_{l=0}\hbar^l J_3^{(l)}$ as a functional of the order $\hbar^0$ quantities ϕ, G and V_3. From (53) and (59), we can see that $J^{(0)}_{3ABC} = -g_{ABC} + V_{3ABC}$ which agrees with (24). $J_3^{(l)}(l \geq 1)$ can be obtained from $\hbar^{l+2}$ term of (59). For example, $J_3^{(1)}$ and $J_3^{(2)}$ can be determined from

$$\left[\frac{\delta^2\Delta^{(2)}}{\delta J_{3PQR}\delta J_{3ABC}}\right]_{J_3=J_3^{(0)}} J^{(1)}_{3ABC} + \left[\frac{\delta\Delta^{(3)}}{\delta J_{3PQR}}\right]_{J_3=J_3^{(0)}} = 0, \tag{60}$$

$$\left[\frac{\delta^2\Delta^{(2)}}{\delta J_{3PQR}\delta J_{3ABC}}\right]_{J_3=J_3^{(0)}} J_{3ABC}^{(2)} + \left[\frac{\delta^2\Delta^{(3)}}{\delta J_{3PQR}\delta J_{3ABC}}\right]_{J_3=J_3^{(0)}} J_{3ABC}^{(1)} + + \left[\frac{\delta\Delta^{(4)}}{\delta J_{3PQR}}\right]_{J_3=J_3^{(0)}} = 0. \qquad (61)$$

and higher orders of $J_3^{(l)}(l \geq 3)$ can be obtained by similar procedure if $J_3^{(k)}(k \prec l)$ are determined. Note that the Feynman diagrams of $\Delta^{(l)}(l \geq 3)$ consist of the propagator G and the three-point vertex $J_3{+}g$ and that $\left[\frac{\delta\Delta^{(l)}}{\delta J_{3PQR}}\right]_{J_3=J_3^{(0)}}$ replaces the three-point vertex of $\frac{\delta\Delta^{(l)}}{\delta J_3}$ by V_3. Then, since $\frac{\delta^2\Delta^{(2)}}{\delta J_{3PQR}\delta J_{3ABC}} = -\frac{1}{36}[\, G_{AP}G_{BQ}G_{CR} + permutations]$, the whole procedure to determine the $J_3^{(l)}(l \geq 1)$ does not depend on ϕ as long as $J_{3ABC}^{(k)}(k \prec l)$ does not depend on ϕ.

---

[1] For a review and references, see M. Sher, Phys. Rep. 179, 273 (1989).
[2] For a review and references, see J. Berges, AIP Conf. Proc. 739, 3 (2005).
[3] J. M. Cornwall, R. Jackiw and E. Toumboulis, Phys. Rev. D10, 2428 (1974).
[4] J. M. Luttinger and J. C. Ward, Phys. Rev. 118, 1417 (1960) ; G. Baym, Phys. Rev. 127, 1361 (1962); G. Baym and L. Kadanoff, Phys. Rev. 124, 287 (1961); C. D. Dominicis and P. C. Martin, J. Math. Phys. 5, 14 (1964); A. N. Vasiliev, Functional Methods in Quantum Field Theory and Statistical Physics (Gordon and Breach Science Publishers, New York, 1998).
[5] J. Berges, Phys. Rev. D70, 105010 (2004).
[6] M. E. Carrington, Eur. Phys. J. C35, 383 (2004) and arXiv: hep-ph/ 0709.0706.
[7] C. Kim, Phys. Rev. D72, 085007 (2005).
[8] M. Bachmann, H. Kleinert, and A. Pelster, Phys.Rev. D61, 085017 (2000).
[9] B. Kastening, Phys. Rev. E 61, 3501 (2000).
[10] H. Kleinert, A. Pelster, B. Kastening, and M. Bachmann , Phys.Rev. E62, 1537 (2000).
[11] A. Pelster, H. Kleinert, and M. Bachmann , Annals Phys. 297, 363 (2002).
[12] H. Kleinert, A. Pelster, B. and Van den Bossche, Physica A312, 141 (2002).
[13] A. Pelster and H. Kleinert, Physica A323, 370 (2003)
[14] A. Pelster and K. Glaum, Physica A335, 455 (2004).
[15] C. Kim, Phys. Rev. D74 067702 (2006).
[16] C. Kim, J. of the Korean Phys. Soc. 51, 1873 (2007).
[17] S. Coleman and E. Weinberg, Phys. Rev. D7, 1888 (1973).
[18] K. Kajantie, M. Laine, and Y. Schroder, Phys.Rev. D65 045008 (2002).

FIGURE CAPTIONS

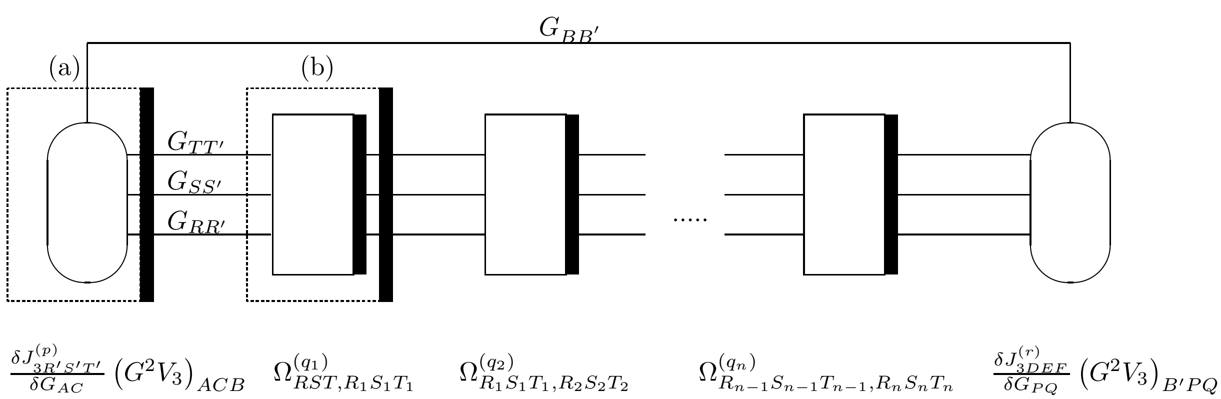


Fig. 1. Graphical representations of the third term of the R.H.S. of Eq.(40).